A Spectral Characterization of Operators Having Rank k

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ABSTRACT

Let $A \in B(X)$, the algebra of all bounded linear operators on a complex Banach space X, and let k be a positive integer. It is proved that A has rank k if and only if (i) $\bigcap_{m=1}^{k+1}\sigma(T + c_m A) \subset \sigma(T)$ for every $T \in B(X)$ and some nonzero scalars $c_1, \dots, c_{k+1}$, and (ii) $\bigcap_{m=1}^{k}\sigma(T' + c'_m A) \not\subset \sigma(T')$ for some $T' \in B(X)$ and some distinct scalars $c'_1, \dots, c'_k$.

Let X be a Banach space over the complex scalar field $\mathbb{C}$, and let $B(X)$ denote the algebra of all bounded linear operators on X. For $T \in B(X)$, let $\sigma(T)$ denote the spectrum of T. The following results were proved in [2]:

(1) Let $A \in B(X)$. Then $A = 0$ if and only if

$$\sigma(T + A) \subset \sigma(T)$$

for every $T \in B(X)$.

(2) Let $A \in B(X)$, $A \neq 0$. Then A has rank 1 if and only if

$$\sigma(T + A) \cap \sigma(T + cA) \subset \sigma(T)$$

for every $T \in B(X)$ and every scalar $c \neq 1$.

The purpose of the present note is to strengthen and to generalize the above results. We prove

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THEOREM 1. *Let $A \in B(X)$, and k be a nonnegative integer.*

(a) *If* $\operatorname{rank} A \leqslant k$, *then*

$$\bigcap_{m=1}^{k+1} \sigma(T + c_m A) \subset \sigma(T)$$

for every $T \in B(X)$ and every set of distinct scalars $c_1, \ldots, c_{k+1}$.

(b) *If*

$$\bigcap_{m=1}^{k+1} \sigma(T + c_m A) \subset \sigma(T)$$

for every $T \in B(X)$ of rank at most $k+1$ and some nonzero scalars $c_1, \ldots, c_{k+1}$, then $\operatorname{rank} A \leqslant k$.

This theorem is proved by modifying the arguments given in [2]. It yields the characterization of operators having rank k stated in the abstract.

We use the notation of [2]. In particular, for u in X and f in the dual X^* of X, we denote by $\langle u, f \rangle$ the value of f at u.

Let k be a positive integer. It is clear that $A \in B(X)$ has rank at most k if and only if there are $x_1, \ldots, x_k$ in X and $f_1, \ldots, f_k$ in X^* such that for all $u \in X$

$$Au = \langle u, f_1 \rangle x_1 + \cdots + \langle u, f_k \rangle x_k. \tag{$*$}$$

LEMMA 2. *For $T \in B(X)$, $\lambda \notin \sigma(T)$, and A given by $(*)$, we have $\lambda \in \sigma(T + A)$ if and only if* $\det[a_{i,j}(\lambda) - \delta_{i,j}] = 0$, *where*

$$a_{i,j}(\lambda) = \langle (\lambda - T)^{-1} x_j, f_i \rangle, \qquad i, j = 1, \ldots, k,$$

and $\delta_{i,j}$ is the Kronecker symbol.

Proof. Let $\det[a_{i,j}(\lambda) - \delta_{i,j}] = 0$. Then there are scalars $t_1, \ldots, t_k$, not all zero, such that

$$\sum_{j=1}^{k} a_{i,j}(\lambda) t_j - t_i = 0, \qquad i = 1, \ldots, k.$$

Let $u = \sum_{j=1}^{k} t_j(\lambda - T)^{-1}x_j$. Since

$$\langle u, f_i \rangle = \sum_{j=1}^{k} t_j a_{i,j}(\lambda) = t_i, \qquad i = 1,\ldots,k,$$

and since $t_i \neq 0$ for some i, we see that $u \neq 0$. Also,

$$\begin{aligned}(T + A)u &= (T - \lambda)u + \lambda u + \sum_{i=1}^{k} \langle u, f_i \rangle x_i \\ &= -\sum_{i=1}^{k} t_i x_i + \lambda u + \sum_{i=1}^{k} t_i x_i \\ &= \lambda u,\end{aligned}$$

so that λ is an eigenvalue of $T + A$.

Conversely, let $\lambda \in \sigma(T + A)$. Since $\lambda \notin \sigma(T)$ and A is compact, we see by the Fredholm alternative that λ is an eigenvalue of $T + A$. Let $0 \neq u \in X$ be a corresponding eigenvector: $(T + A)u = \lambda u$. Therefore,

$$u = \sum_{j=1}^{k} \langle u, f_j \rangle (\lambda - T)^{-1} x_j.$$

It follows that

$$\langle u, f_i \rangle = \sum_{j=1}^{k} \langle u, f_j \rangle a_{i,j}(\lambda), \qquad i = 1,\ldots,k.$$

Since $u \neq 0$, not all of $\langle u, f_1 \rangle, \ldots, \langle u, f_k \rangle$ can be zero. Hence $\det[a_{i,j}(\lambda) - \delta_{i,j}] = 0$. ∎

Proof of Theorem 1(a). Let $A \in B(X)$ and $\operatorname{rank} A \leqslant k < \infty$. If $k = 0$, then $A = 0$, and $\sigma(T + c_1 A) = \sigma(T)$ for every $T \in B(X)$ and every scalar c_1. Now assume that $1 \leqslant k$. Then A is given by $(*)$. Let $T \in B(X)$, $c_1, \ldots, c_{k+1}$ be distinct scalars, and $\lambda \notin \sigma(T)$. Were $\lambda \in \sigma(T + c_m A)$ for every $m = 1, \ldots, k+1$, then by Lemma 2,

$$\det\left[c_m a_{i,j}(\lambda) - \delta_{i,j}\right] = 0$$

for every $m = 1, \ldots, k+1$. However, $\det[xa_{i,j}(\lambda) - \delta_{i,j}]$ is a polynomial in x of degree at most k, and cannot have $k+1$ distinct roots. Hence $\lambda \notin \bigcap_{m=1}^{k+1} \sigma(T + c_m A)$. ■

Lemma 3. *Let $\lambda \in \mathbb{C}$. Let $D \in B(X)$ be such that $D^l = 0$ but $D^{l-1} \neq 0$, where l is a positive integer, and $A = \lambda + D$. Let m be a positive integer and assume that* rank $A \geqslant m$. *Then there are closed subspaces $Y_1, \ldots, Y_q$ and Z of X which satisfy the following conditions*:

(i) $X = Y_1 \oplus \cdots \oplus Y_q \oplus Z$;

(ii) *for* $j = 1, \ldots, q$, $\dim Y_j = n_j$;

(iii) Y_j *is invariant under* A, *and, in a suitable basis, the operator* $A|_{Y_j}$ *is represented by the* $n_j \times n_j$ *Jordan block*

$$\begin{bmatrix} \lambda & & & & \bigcirc \\ 1 & \cdot & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ \bigcirc & & & 1 & \lambda \end{bmatrix},$$

where $l = n_1 \geqslant \cdots \geqslant n_q \geqslant 1$;

(iv) $\sum_{j \leqslant q} \operatorname{rank}(A|_{Y_j}) \geqslant m > \sum_{j < q} \operatorname{rank}(A|_{Y_j})$.

Proof. First, let $l = 1$. Then $D = 0$, $A = \lambda$, and $\dim X = \operatorname{rank} A \geqslant m$. Let $\{y_1, \ldots, y_m\}$ be a linearly independent set in X, and $Y_j = \operatorname{span}\{y_j\}$, $j = 1, \ldots, m$, so that $\operatorname{rank}(A|_{Y_j}) = 1$. Let $q = m$, and note that conditions (ii), (iii), and (iv) hold. Next, let $l > 1$. Since $D^{l-1} \neq 0$, let $y_1 \in X$ be such that $D^{l-1} y_1 \neq 0$. Then the set $\{y_1, Dy_1, \ldots, D^{l-1} y_1\}$ is linearly independent. Let $Y_1 = \operatorname{span}\{y_1, Dy_1, \ldots, D^{l-1} y_1\}$, so that Y_1 is invariant under D and $\dim Y_1 = l$. The matrix representation of $A|_{Y_1}$ with respect to the ordered basis $y_1, \ldots, D^{l-1} y_1$ is

$$\begin{bmatrix} \lambda & & & & \bigcirc \\ 1 & \cdot & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ \bigcirc & & & 1 & \lambda \end{bmatrix}.$$

Also, there is a subspace W_1 of X which is invariant under D, and $X = Y_1 + W_1$, $Y_1 \cap W_1 = \{0\}$. (Cf. Lemma 6.5.4 of [1]; in case X is infinite dimensional, Zorn's lemma can be employed.) Let $n_1 = l$, and note that

$$\operatorname{rank} A = \operatorname{rank}(A|_{Y_1}) + \operatorname{rank}(A|_{W_1}) \geqslant m.$$

If $\operatorname{rank}(A|_{Y_1}) \geqslant m$, then we let $q = 1$. If $\operatorname{rank}(A|_{Y_1}) < m$, then $\operatorname{rank}(A|_{W_1}) \geqslant 1$ and there is an integer n_2, $n_1 \geqslant n_2 \geqslant 1$, such that $(D|_{W_1})^{n_2} = 0$, but $(D|_{W_1})^{n_2 - 1} \neq 0$. We repeat the above argument with X replaced by W_1, and n_1 by n_2. In a finite number, say q, of such steps, we obtain subspaces $Y_1, \dots, Y_q$ of X such that $Y_i \cap Y_j = \{0\}$ for $i \neq j$, and conditions (ii), (iii), and (iv) hold.

For any $l \geqslant 1$, the finite dimensional subspace $Y = Y_1 \oplus \cdots \oplus Y_q$ is closed in X, and there is a closed subspace Z of X such that $X = Y \oplus Z$, i.e., condition (i) holds as well. ■

LEMMA 4. *Let N be an $n \times n$ Jordan block, p be a positive integer with $p \leqslant \operatorname{rank} N$, and $\alpha, \beta_1, \dots, \beta_p$ be nonzero scalars. Then there is an $n \times n$ matrix M such that $\operatorname{rank} M \leqslant p$, and $\alpha \notin \sigma(M)$, but $\alpha \in \sigma(M + \beta_s N)$ for $s = 1, \dots, p$.*

Proof. Consider

$$N = \begin{bmatrix} \lambda & & & & 0 \\ 1 & \ddots & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ 0 & & & 1 & \lambda \end{bmatrix},$$

where $\lambda \in \mathbb{C}$. If $\lambda \neq 0$, then $\operatorname{rank} N = n$ and we can let

$$M = \operatorname{diag}(\alpha - \lambda\beta_1, \dots, \alpha - \lambda\beta_p, 0, \dots, 0).$$

If $\lambda = 0$, then $\operatorname{rank} N = n - 1$ and we can let

$$M = \left[\begin{array}{cccc|c|c} \alpha & & & 0 & \alpha & \\ -\beta_1 & \alpha & & & -\beta_1 & \\ & -\beta_2 & \ddots & & 0 & 0 \\ & & \ddots & \alpha & \vdots & \\ 0 & & & -\beta_p & 0 & \\ \hline & & 0 & & & 0 \end{array}\right].$$ ■

Proof of Theorem 1(b). Let $A \in B(X)$, and $c_1, \ldots, c_{k+1}$ be nonzero scalars such that

$$\bigcap_{m=1}^{k+1} \sigma(T + c_m A) \subset \sigma(T)$$

for every $T \in B(X)$ of rank at most $k+1$. Let α be a nonzero scalar.

First we show that for every x in X, the set $\{x, Ax, \ldots, A^{k+1}x\}$ is linearly dependent. Suppose, to the contrary, there were $u \in X$ for which the set $\{u, Au, \ldots, A^{k+1}u\}$ was linearly independent. Let $U = \operatorname{span}\{u, Au, \ldots, A^{k+1}u\}$, and V be a closed subspace of X such that $X = U \oplus V$. Define an operator $T \in B(X)$ by

$$\begin{aligned} Tu &= \alpha u - c_1 Au, \\ TAu &= \alpha Au - c_2 A^2 u, \\ &\vdots \\ TA^k u &= \alpha A^k u - c_{k+1} A^{k+1} u, \\ TA^{k+1} u &= \alpha u - c_1 Au, \\ Tv &= 0 \qquad \text{for} \quad v \in V. \end{aligned}$$

Then $(T + c_j A)A^{j-1}u = \alpha A^{j-1}u$, so that α is an eigenvalue of $T + c_j A$ for $j = 1, \ldots, k+1$. However, $T(u - A^{k+1}u) = 0$, so that $\operatorname{rank} T = k+1$, and

$$\det\left[(T - \alpha)|_U\right] = \alpha c_1 \cdots c_{k+1} \neq 0,$$

so that $\alpha \notin \sigma(T|_U) \cup \sigma(T|_V) = \sigma(T)$. This contradicts the assumption

$$\bigcap_{m=1}^{k+1} \sigma(T + c_m A) \subset \sigma(T).$$

Hence there is a nonnegative integer $k_0 \leqslant k$ such that for every $x \in X$, the set $\{x, Ax, \ldots, A^{k_0+1}x\}$ is linearly dependent, but the set $\{u, Au, \ldots, A^{k_0}u\}$ is linearly independent for some $u \in X$, provided $X \neq \{0\}$. Note that $k_0 = 0$ if and only if A is a scalar multiple of the identity operator.

By an argument exactly similar to the one given in the last paragraph on p. 257 of [2], it follows that there is a polynomial p of degree k_0+1 with complex coefficients such that $p(A)=0$. Let p_0 be the (unique) monic polynomial of the smallest degree such that $p_0(A)=0$. Let

$$p_0(t)=(t-\lambda_1)^{l_1}\cdots(t-\lambda_n)^{l_n},$$

where $\lambda_i\neq\lambda_j$ for $i\neq j$ and $1\leqslant l_i<\infty$ for $i,j=1,\dots,n$. It is easy to see that

$$\sigma(A)=\{\lambda_1,\dots,\lambda_n\}$$

and that each λ_i is an eigenvalue of A.

Let P_i denote the spectral projection associated with A and λ_i, i.e.,

$$P_i=\frac{1}{2\pi i}\int_{\Gamma_i}(z-A)^{-1}\,dz, \qquad i=1,\dots,n,$$

where Γ_i is a circle which encloses λ_i but no other point of $\sigma(A)$ (cf. [3, pp. 178–181]). Then

$$P_iP_j=\delta_{i,j}P_i, \quad i,j=1,\dots,n, \qquad P_1+\cdots+P_n=I.$$

Let X_i denote the range of P_i. If we let $D_i=(A-\lambda_i)P_i$, then

$$A|_{X_i}=\lambda_i+D_i|_{X_i},$$

where $(D_i|_{X_i})^{l_i}=0$, but $(D_i|_{X_i})^{l_i-1}\neq 0$. Let $k_i=\operatorname{rank}(A|_{X_i})$. Since

$$A=A|_{X_1}\oplus\cdots\oplus A|_{X_n},$$

we see that $\operatorname{rank} A=\sum_{i=1}^n k_i$.

To prove $\operatorname{rank} A\leqslant k$, we assume that $\sum_{i=1}^n k_i\geqslant k+1$ and arrive at a contradiction. First we find r such that $1\leqslant r\leqslant n$ and

$$\sum_{i\leqslant r}k_i\geqslant k+1>\sum_{i<r}k_i. \qquad (**)$$

Let $i < r$. Letting $m = k_i$ in Lemma 3, we see that

$$X_i = Y_{i,1} \oplus \cdots \oplus Y_{i,q_i} \oplus Z_i,$$

and for $j = 1, \ldots, q_i$, the operator $A|_{Y_{i,j}}$ is represented in a suitable basis by the $n_{i,j} \times n_{i,j}$ Jordan block

$$N_{i,j} = \begin{bmatrix} \lambda_i & & & & \bigcirc \\ 1 & \cdot & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ \bigcirc & & & 1 & \lambda_i \end{bmatrix},$$

where $l_i \geqslant n_{i,1} \geqslant \cdots \geqslant n_{i,q_i} \geqslant 1$ and

$$\sum_{j \leqslant q_i} \operatorname{rank}(A|_{Y_{i,j}}) \geqslant k_i > \sum_{j < q_i} \operatorname{rank}(A|_{Y_{i,j}}).$$

Next, consider $i = r$. If we let

$$m = k + 1 - \sum_{i < r} k_i,$$

then it follows from $(**)$ that $k_r \geqslant m \geqslant 1$. Again by Lemma 3,

$$X_r = Y_{r,1} \oplus \cdots \oplus Y_{r,q_r} \oplus Z_r,$$

and for $j = 1, \ldots, q_r$, the operator $A|_{Y_{r,j}}$ is represented in a suitable basis by the $n_{r,j} \times n_{r,j}$ Jordan block

$$N_{r,j} = \begin{bmatrix} \lambda_r & & & & \bigcirc \\ 1 & \cdot & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ \bigcirc & & & 1 & \lambda_r \end{bmatrix},$$

where $l_r \geqslant n_{r,1} \geqslant \cdots \geqslant n_{r,q_r} \geqslant 1$ and

$$\sum_{j \leqslant q_r} \operatorname{rank}\left(A|_{Y_{r,j}}\right) \geqslant k+1-\sum_{i<r} k_i > \sum_{j<q_r} \operatorname{rank}\left(A|_{Y_{r,j}}\right). \qquad (***)$$

For $i<r$ and $1 \leqslant j \leqslant q_i$ as well as for $i=r$ and $j<q_r$, let

$$p_{i,j} = \operatorname{rank} N_{i,j} = \operatorname{rank}\left(A|_{Y_{i,j}}\right).$$

Also, let

$$p_{r,q_r} = k+1-\sum_{i<r}\sum_{j \leqslant q_i} p_{i,j} - \sum_{j<q_r} p_{r,j}.$$

Since $\sum_{j \leqslant q_i} p_{i,j} = k_i$ for $i<r$, it follows from $(***)$ that

$$1 \leqslant p_{r,q_r} \leqslant \operatorname{rank}\left(A|_{Y_{r,q_r}}\right) = \operatorname{rank} N_{r,q_r}.$$

Let, now, $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant q_i$. If $\beta_{i,j,s}$, $1 \leqslant s \leqslant p_{i,j}$, are nonzero scalars, Lemma 4 shows that there is an $n_{i,j} \times n_{i,j}$ matrix $M_{i,j}$ such that $\operatorname{rank} M_{i,j} \leqslant p_{i,j}$, and $\alpha \in \sigma(M_{i,j})$ but $\alpha \notin \sigma(M_{i,j} + \beta_{i,j,s} N_{i,j})$. Since $\sum_{i=1}^{r}\sum_{j=1}^{q_i} k_{i,j} = k+1$, we let

$$\{\beta_{i,j,s} : 1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant q_i,\ 1 \leqslant s \leqslant p_{i,j}\} = \{c_m : 1 \leqslant m \leqslant k+1\}.$$

Define an operator $T \in B(X)$ as follows. For $i=1,\ldots,r$ and $j=1,\ldots,q_i$, $T|_{Y_{i,j}}$ is represented by the matrix $M_{i,j}$ in the basis for $Y_{i,j}$ chosen earlier; for $i=1,\ldots,r$, $T|_{Z_i}=0$; and for $i=r+1,\ldots,n$, $T|_{X_i}=0$. Then

$$\operatorname{rank} T = \sum_{i=1}^{r}\sum_{j=1}^{q_i} \operatorname{rank} M_{i,j}$$

$$\leqslant \sum_{i=1}^{r}\sum_{j=1}^{q_i} p_{i,j} = k+1,$$

and $\alpha \notin \sigma(T)$, but $\alpha \in \sigma(T + c_m A)$ for $m=1,\ldots,k+1$. This contradicts the assumption

$$\bigcap_{m=1}^{k+1} \sigma(T + c_m A) \subset \sigma(T). \qquad \blacksquare$$

REMARK 5. The above proof of Theorem 1(b) shows, in fact, that if *some* nonzero scalar α belongs to $\sigma(T)$ whenever it belongs to $\bigcap_{m=1}^{k+1}\sigma(T + c_m A)$ for every $T \in B(X)$ of rank at most $k+1$ and some nonzero scalars $c_1, \dots, c_{k+1}$, then rank $A \leqslant k$. This statement is stronger than the statement of Theorem 1(b).

REMARK 6. Letting $k = 0$ and 1 in Theorem 1(b), we have the following results:

(1′) *If* $\sigma(T + c_1 A) \subset \sigma(T)$ *for every* $T \in B(X)$ *of rank at most* 1 *and some nonzero scalar* c_1, *then* $A = 0$.

(2′) *If* $A \neq 0$ *and if* $\sigma(T + c_1 A) \cap \sigma(T + c_2 A) \subset \sigma(T)$ *for every* $T \in B(X)$ *of rank at most* 2 *and some nonzero scalars* c_1 *and* c_2, *then* rank $A = 1$.

It is clear that these results strengthen results (1) and (2) mentioned in the beginning of this note. Although the result (1′) follows easily from the proof of result (1) given in [2], result (2′) is not an immediate consequence of the proof of result (2) given in [2]. Of course, the main point of our Theorem 1 is to generalize results (1) and (2) to operators of rank $k > 1$.

The author thanks the referees for suggesting modifications of Lemma 3 and Lemma 4.

Note added in proof: For operators of rank 1 on a finite dimensional space X, a sharpening of result (2′) stated above is given in "On a spectral characterization of rank one matrices" by H. Deguang and the author (page 1, this issue). It will be of interest to sharpen Theorem 1 similarly for operators of rank $k > 1$ on a finite dimensional space X.

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